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Technical Notes

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Asymptotic Suction Flow with Partial Anisotropic Slip

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Introduction

VISCOUS flow with partial slip occurs in a variety of situations such as the slip regime of rarefied gases [1–3]. It also models the boundary of rough or grooved surfaces [4]. However, exact solutions of the viscous flow with slip are even rarer than those with zero slip [5], due to the fact that conformal transformation fails for partial slip boundary conditions. In this Note we present simple exact solutions of the flow over a plate and a cylinder with suction and anisotropic partial slip. The anisotropy may be caused by parallel slits on the surface through which suction occurs. If there were no slip, the solution reduces to the exact asymptotic suction flow found by Griffith and Meredith [6,7] important in airfoil boundary layer control.

The no-slip boundary condition on a rough or striated surface is very difficult to enforce. Navier first suggested the boundary condition.

$$u = N\tau \tag{1}$$

to replace the rough surface by a smooth surface with partial slip. Here u is the tangential slip velocity, τ is the shear stress, and N is the slip coefficient. The geometry of the roughness is reflected in N. There is no slip when $N = 0$ and total slip when $N \to \infty$. In this Note we investigate partial slip where N is nonzero and finite. The theoretical determination of the slip coefficient can sometimes be obtained from microfluid dynamical study of the roughness. Because the local (pore scale) Reynolds number is indeed small, the slip coefficients were determined under the Stokes assumption for surfaces composed of parallel, semi-infinite fins [8–11] parallel thin slats in the same plane [12] and parallel grooves of infinite or finite depths [4,13]. It was found that the slip coefficient is a function of the roughness geometry, especially amplitude and wavelength [4]. Also, the slip coefficient is directional, being larger along the striations than that across the striations. Because Stokes flow is linear, one can also superpose a suction velocity normal to the mean surface without changing the tangential slip coefficients.

In this Note we replaced a striated rough surface by a smooth surface with anisotropic partial slip coefficients. Two simple 3-D asymptotic suction solutions are presented. These are also rare exact solutions of the Navier–Stokes equations.

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Asymptotic Suction Slip Flow over a Plate

Figure 1a shows a fluid of velocity U moving in the x direction parallel to a porous plate with suction velocity W. The top view (Fig. 1b) shows the plate has grooves or striations inclined at an angle α with the y axis. Let (u, υ, w) be the velocity components in the Cartesian (x, y, z) directions. Let (u_1, u_2) be the velocity components across and along the striations, respectively. Then

$$u_1 = u\cos\alpha - \upsilon\sin\alpha, \qquad u_2 = u\sin\alpha + \upsilon\cos\alpha \tag{2}$$

For asymptotic suction flow the diffusion of vorticity from the surface is balanced by the transport through suction. Thus all variables are functions of z only and are independent of x or y (e.g., Sherman [7]). Continuity equation shows $w = -W$. The Navier–Stokes equations reduce to

$$-W\frac{du}{dz} = \nu\frac{d^2u}{dz^2} \tag{3}$$

$$-W\frac{d\upsilon}{dz} = \nu\frac{d^2\upsilon}{dz^2} \tag{4}$$

where ν is the kinematic viscosity. Equations (3) and (4) are coupled through the boundary conditions. Far from the plate

$$u(\infty) = U, \qquad \upsilon(\infty) = 0 \tag{5}$$

and on the plate

$$u_1(0) = N_1\rho\nu\frac{du_1}{dz}(0) \tag{6}$$

$$u_2(0) = N_2\rho\nu\frac{du_2}{dz}(0) \tag{7}$$

Here ρ is the density and N_1, N_2 are the slip coefficients across and along the striations, respectively. Equations (3–5) yield

$$u = U(1 - Ae^{-Wz/\nu}) \tag{8}$$

$$\upsilon = UBe^{-Wz/\nu} \tag{9}$$

Where A, B are constants to be determined from Eqs. (2) and (6–9). The result is

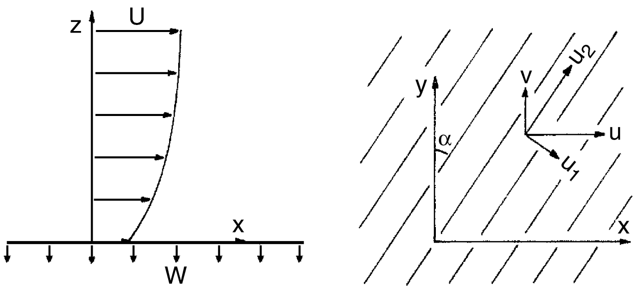
$$A = \frac{1 + \lambda_2 - (\lambda_2 - \lambda_1)\sin^2\alpha}{(1 + \lambda_1)(1 + \lambda_2)} \tag{10}$$

$$B = \frac{(\lambda_2 - \lambda_1)\cos\alpha\sin\alpha}{(1 + \lambda_1)(1 + \lambda_2)} \tag{11}$$

where

$$\lambda_1 = N_1\rho W, \qquad \lambda_2 = N_2\rho W \tag{12}$$

are nondimensional slip coefficients. For an isotropic or uniformly rough surface, $\lambda_1 = \lambda_2$ and the velocity in the y direction would be zero. If furthermore the slip is zero, $A = 1$ and $B = 0$ and the solution reduces to the well known asymptotic suction solution. In



a) b)

Fig. 1 a) Flow past a plate with suction; b) top view of striations and velocity components.

nonisotropic cases, $\lambda_2 > \lambda_1 > 0$. The forces per area experienced by the plate are ρAWU and $-\rho BWU$ in the x, y directions. Notice the maximum force along the free stream occurs at $\alpha = 0$ deg and maximum force transverse to the free stream occurs at an angle of $\alpha = 45$ deg.

Asymptotic Suction Slip Flow Along a Cylinder

Consider the flow along a long circular cylinder of radius a. The cylinder is striated and has suction velocity W on its porous surface. Let (u, υ, w) be the velocity components in the cylindrical (z, θ, r) directions, respectively. If α is the angle of inclination of the striations to the r, θ plane, Eq. (2) holds. Using symmetry, all variable are functions or r and the continuity equation gives

$$w = -\frac{aW}{r} \tag{13}$$

The Navier–Stokes equations yield

$$w\frac{du}{dr} = \nu\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) \tag{14}$$

$$w\frac{d\upsilon}{dr} + \frac{w\upsilon}{r} = \nu\left(\frac{d^2\upsilon}{dr^2} + \frac{1}{r}\frac{d\upsilon}{dr} - \frac{\upsilon}{r^2}\right) \tag{15}$$

$$w\frac{dw}{dr} - \frac{\upsilon^2}{r} = -\frac{1}{\rho}\frac{dp}{dr} + \nu\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - \frac{w}{r^2}\right) \tag{16}$$

where p is the pressure. The solution satisfying

$$u(\infty) = U, \qquad \upsilon(\infty) = 0 \tag{17}$$

is

$$u = U\left[1 - A\left(\frac{a}{r}\right)^s\right] \tag{18}$$

$$\upsilon = U\left[C\frac{a}{r} + B\left(\frac{a}{r}\right)^{s-1}\right] \tag{19}$$

where $s = Wa/\nu$. If further circulation at infinity is to be zero, $C = 0$ and $s > 2$. The pressure is then integrated from Eq. (16)

$$p = p_\infty - \rho\left[\frac{W^2a^2}{2r^2} + \frac{B^2r^{-2(s-1)}}{2(s-1)}\right] \tag{20}$$

The slip boundary conditions are

$$u_1(a) = N_1\tau_1(a) = N_1\rho\nu\left[\cos\alpha\frac{du}{dr} - \sin\alpha\left(\frac{d\upsilon}{dr} - \frac{\upsilon}{r}\right)\right]_{r=a} \tag{21}$$

$$u_2(a) = N_2\tau_2(a) = N_2\rho\nu\left[\sin\alpha\frac{du}{dr} + \cos\alpha\left(\frac{d\upsilon}{dr} - \frac{\upsilon}{r}\right)\right]_{r=a} \tag{22}$$

Equation (2) then yields the coefficients A and B, which are found to be exactly the same as those given in Eqs. (10) and (11). The longitudinal force per length of cylinder is $2\pi a\rho AWU$ and the torque per length is $2\pi a^2\rho BWU/(1-s)$. Notice that when there is no slip, $A = 1$ and $B = 0$ and our solution reduces to those of Wuest [14] and Lew [15]. In such a case, the restriction that $s > 2$ does not apply.

Conclusions

Exact solutions of the Navier–Stokes equations are important but extremely rare [5]. This Note presents two simple, closed-form, three-dimensional exact solutions of the Navier–Stokes equations. The results are useful and can be applied to boundary layer control through minute slits on a wall. The flow is dependent on the nondimensional slip coefficients λ_1 and λ_2, which are functions of slit geometry and suction velocity W.

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